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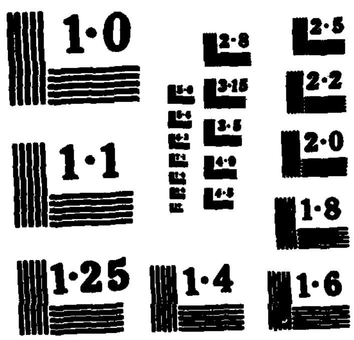

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EXISTENCE OF EQUILIBRIUM PRICES FOR A
SIMPLE PLANNING MODEL

Hui Hu

TECHNICAL REPORT SOL 85-10

June 1985

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EXISTENCE OF EQUILIBRIUM PRICES FOR A SIMPLE PLANNING MODEL

Hui Hu

Abstract

Consider parametric LP:

$$\begin{array}{lll} \min & -\theta & \\ \text{s.t.} & AY + \theta(-d^0 + M\hat{\pi}) \geq b & \\ & 0 \leq Y \leq K \quad , \quad \theta \geq 0 & \end{array} \tag{I}$$

where M is a positive definite (not necessarily symmetric) matrix, $K > 0$, $\hat{\pi}$ is a parameter, $\hat{\pi} \in S = \{\pi \geq 0: e\pi = 1\}$.

For each fixed $\hat{\pi} \in S$, we can solve (I) and it's dual problem and get optimal θ^*, Y^*, σ^*, π^*. The question is, is there a $\hat{\pi} \in S$ such that after solving the corresponding LP and normalizing the dual price π^*, it turns out that $\pi^*/e\pi^* = \hat{\pi}$? In this paper, we are going to show that under certain conditions, such a $\hat{\pi}$ does exist.

The economic interpretation of the above model will be given at the end of the paper.

EXISTENCE OF EQUILIBRIUM PRICES FOR A SIMPLE PLANNING MODEL

Hui Hu

Notation:

$$(P\hat{\pi}): \quad \begin{aligned} \min \quad & (0, 0, \ldots, 0, -1) \begin{bmatrix} Y \\ \theta \end{bmatrix} \\ \text{s.t.} \quad & \begin{bmatrix} -I & 0 \\ A & -d^0 + M\hat{\pi} \end{bmatrix} \begin{bmatrix} Y \\ \theta \end{bmatrix} \geq \begin{bmatrix} -K \\ b \end{bmatrix} \\ & Y \geq 0, \quad \theta \geq 0 \end{aligned}$$

which is equivalent to:

$$\begin{aligned} \min \quad & -\theta \\ \text{s.t.} \quad & AY + \theta(-d^0 + M\hat{\pi}) \geq b \\ & 0 \leq Y \leq K, \ \theta \geq 0 \end{aligned}$$

The corresponding dual problem is:

$$(D\hat{\pi}): \quad \begin{aligned} \max \quad & (\sigma, \pi) \begin{bmatrix} -K \\ b \end{bmatrix} \\ \text{s.t.} \quad & (\sigma, \pi) \begin{bmatrix} -I & 0 \\ A & -d^0 + M\hat{\pi} \end{bmatrix} \leq (0, 0, \ldots, 0, -1) \\ & \sigma \geq 0, \ \pi \geq 0 \end{aligned}$$

which is equivalent to:

$$\max \quad -\sigma K + \pi b$$

$$\text{s.t.} \quad \begin{aligned} \pi A &\leq \sigma \\ \pi(-d^0 + M\hat{\pi}) &\leq -1 \\ \pi \geq 0, \quad \sigma &\geq 0 \end{aligned}$$

Let $F(P\hat{\pi})$ and $F(D\hat{\pi})$ denote the feasible regions of $(P\hat{\pi})$ and $(D\hat{\pi})$ respectively. $S = \{\pi \geq 0 : e\pi = 1\}$.

Definition. Let $D \subseteq R^n$, $U \subseteq R^m$, a point to set map $f : D \to P(U)$ is upper hemicontinuous at $\bar{x} \in D$ if for all $x^k \to \bar{x}$ and $y^k \in f(x^k)$ such that $y^k \to \bar{y}$, we have $\bar{y} \in f(\bar{x})$. If f is upper hemicontinuous at all $x \in D$, f is called upper hemicontinuous.

Lemma 1. If $\{Y : AY \geq b,\ 0 \leq Y \leq K\} \neq \phi$ and for all $\hat{\pi} \in S$, $d^0 - M\hat{\pi} > 0$, then for any $\hat{\pi} \in S$, $(P\hat{\pi})$ and $(D\hat{\pi})$ have optimal solutions.

Proof. From the assumption we know there exists $\bar{Y} \in \{Y : AY \geq b,\ 0 \leq Y \leq K\}$, therefore

$$\begin{bmatrix} \bar{Y} \\ 0 \end{bmatrix} \in \left\{ \begin{bmatrix} Y \\ \theta \end{bmatrix} \geq 0 : \begin{bmatrix} -1 & 0 \\ A & -d^0 + M\hat{\pi} \end{bmatrix} \begin{bmatrix} Y \\ \theta \end{bmatrix} \geq \begin{bmatrix} -K \\ b \end{bmatrix} \right\}$$

which implies that for any $\hat{\pi} \in S$, $(P\hat{\pi})$ is feasible.

On the other hand, we know that $\{x \geq 0 \colon A\,x \geq b\}$ is bounded if and only if $\{x \geq 0 : Ax \geq 0\} = \{0\}$.

$$\left\{ \begin{bmatrix} Y \\ \theta \end{bmatrix} \geq 0 \colon \begin{bmatrix} -I & 0 \\ A & -d^0 + M\hat{\pi} \end{bmatrix} \begin{bmatrix} Y \\ \theta \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} Y \\ \theta \end{bmatrix} > 0 : \begin{array}{c} Y = 0 \\ (-d^0 + M\hat{\pi})\,\theta > 0 \end{array} \right\} = \{0\} \quad ,$$

therefore for each $\hat{\pi} \in S$, $F(P\hat{\pi})$ is bounded, therefore for all $\hat{\pi} \in S$, $(P\hat{\pi})$ and $(D\hat{\pi})$ have optimal solutions. Q.E.D.

Now define a point to set mapping f: For all $\hat{\pi} \in S$,

$$f(\hat{\pi}) = \{\pi : (\sigma, \pi) \text{ is optimal solution of } (D\hat{\pi})\} \;.$$

Under the assumption of Lemma 1, f is well defined on S, i.e., for all $\hat{\pi} \in S$, $f(\hat{\pi}) \neq \phi$, and furthermore, $f(\hat{\pi})$ is a convex set (that is because $f(\hat{\pi})$ is the projection of a convex set into a lower dimension).

Normalizing f, we get another point to set mapping $\bar{f}$: for all $\hat{\pi} \in S : \bar{f}(\hat{\pi}) = \{\pi/e\pi \colon \pi \in f(\hat{\pi})\}$.

It is easy to see that under the assumption of Lemma 1, for all $\hat{\pi} \in S$, $\bar{f}(\hat{\pi}) \neq \phi$, and $\bar{f}(\hat{\pi})$ is a convex subset of S.

Using these definitions, our question becomes: is there a fixed point of $\bar{f}$?

<u>Theorem 1</u>. If for all $\hat{\pi} \in S$, $d^0 - M\hat{\pi} \geq \bar{d} > 0$ and $\{Y : AY > b, 0 \leq Y \leq K\} \neq \phi$ then there exists $\hat{\pi} \in S$ such that $\hat{\pi} \in \bar{f}(\hat{\pi})$.

Proof. First we prove f is upper hemicontinuous and then use this to prove $\bar{f}$ is upper hemicontinuous; finally we apply the famous Kakutani fixed point theorem to show the existence of fixed point of $\bar{f}$.

For all $\hat{\pi}^i \in S$, $i = 1, 2, \ldots$ and $\hat{\pi}^i \to \hat{\pi}$, for all $\pi^{*i} \in f(\hat{\pi}^i)$ and $\pi^{*i} \to \pi^*$, we show that $\pi^* \in f(\hat{\pi})$.

By our definition of f, for all $\pi^{*i} \in f(\hat{\pi}^i)$, there exists σ^{*i}, such that (σ^{*i}, π^{*i}) is an optimal solution of $(D\hat{\pi}^i)$. From strong duality theorem of linear programming, there exists (Y^{*i}, θ^{*i}), an optimal solution of $(P\hat{\pi}^i)$, such that

$$\pi^{*i}b - \sigma^{*i}K = -\theta^{*i} . \tag{1}$$

Since (Y^{*i}, θ^{*i}) is feasible for $(P\hat{\pi}^i)$, $\theta^{*i}(d^0 - M\hat{\pi}^i) \leq AY^{*i} - b$, $i = 1, 2, \ldots$ and by assumption, $d^0 - M\hat{\pi}^i \geq \bar{d} > 0$ $i = 1, 2, \ldots$, $0 \leq Y \leq K$, we conclude that $\{\theta^{*i}, i = 1, 2, \ldots\}$ is bounded. Combine this with (1), we know $\{\sigma^{*i}k, i = 1, 2, \ldots\}$ is bounded. Since $K > 0$, $\sigma^{*i} \geq 0$, this implies $\{\sigma^{*i}, i = 1, 2, \ldots\}$ is bounded. Therefore there exists a subsequence σ^{*i_j} converges to σ^*. Without loss of generality, assume $\sigma^{*i} \to \sigma^*$, then from (1), we know that $\theta^{*i} \to \theta^*$, and

$$\pi^*b - \sigma^*K = -\theta^* . \tag{2}$$

Since (σ^{*i}, π^{*i}) is feasible for $(D\hat{\pi}^i)$, we have: $\pi^{*i}A \leq \sigma^{*i}$, $\pi^{*i}(-d^0 + M\hat{\pi}^i) \leq -1$, $\pi^{*i} \geq 0$, $\sigma^{*i} \geq 0$. Letting $i \to \infty$ we get $\pi^*A \leq \sigma^*$, $\pi^*(-d^0 + M\hat{\pi}) \leq -1$, $\pi^* \geq 0$, $\sigma^* \geq 0$, i.e., (σ^*, π^*) is feasible for $(D\hat{\pi})$. Similarily, we can assume $Y^{*i} \to Y^*$ and (Y^*, θ^*) is feasible

for $(P\hat{\pi})$. Because (2) holds, by weak duality theorem of linear programming, (Y^*, θ^*) is an optimal solution of $(P\hat{\pi})$, and (σ^*, π^*) is an optimal solution of $(D\hat{\pi})$, therefore, $\pi^* \epsilon f(\hat{\pi})$, f is upper hemicontinuous.

Next, we prove $\bar{f}$ is upper hemicontinuous.

For all $\hat{\pi}^i \epsilon S$, $i = 1, 2, \ldots$ and $\hat{\pi}^i \to \hat{\pi}$, for all $(\pi^{*i}/e\pi^{*i}) \epsilon \bar{f}(\hat{\pi}_i)$ and $(\pi^{*i}/e\pi^{*i}) \to \pi^*$, by assumption $\{Y : AY > b, 0 \leq Y \leq K\} \neq \phi$, we know $\theta^{*i} > 0$, $i = 1, 2, \ldots$, therefore by complement slackness, $\pi^{*i}(d^0 - M\hat{\pi}^i) = 1$, $i = 1, 2, \ldots$. Since we assume $d^0 - M\hat{\pi}^i \geq \bar{d} > 0$, $i = 1, 2, \ldots$, we know $\{\pi^{*i}, i = 1, 2, \ldots\}$ is bounded. Without loss of generality, assume $\pi^{*i} \to \bar{\pi}$, then

$$\hat{\pi}^{*i}/e\pi^{*i} \to \bar{\pi}/e\bar{\pi} = \pi^*, \ \bar{\pi} = e\bar{\pi}\cdot\pi^*.$$

Since we have already shown that f is upper hemicontinuous, we know $\bar{\pi} = e\bar{\pi}\cdot\pi^* \epsilon f(\hat{\pi})$, but $e\pi^* = 1$, therefore $\pi^* \epsilon \bar{f}(\hat{\pi})$, therefore, $\bar{f}$ is upper hemicontinuous.

Under the assumption of Theorem 1, the assumption of Lemma 1 still holds, so for all $\hat{\pi} \epsilon S$, $\bar{f}(\hat{\pi})$ is a nonempty convex subset of S, and we have proved $\bar{f}$ is upper hemicontinuous, therefore by Kakutani Fixed Point Theorem, $\bar{f}$ has a fixed point. Q.E.D.

Next we weaken the assumption of Theorem 1, and prove the existance of a fixed point of $\bar{f}$.

<u>Theorem 2</u>. If $\{Y : AY \geq b, 0 \leq Y \leq K\} \neq \phi$, and for all $\hat{\pi} \epsilon S$, $d^0 - M\hat{\pi} > 0$, $\|d^0 - M\hat{\pi}\| > \varepsilon > 0$, then there exists a $\hat{\pi} \epsilon S$ such that $\hat{\pi} \epsilon \bar{f}(\hat{\pi})$.

Proof. Since the assumption of Lemma 1 still holds, we know that for all $\hat{\pi} \in S$, $f(\hat{\pi}) \neq \phi$ and $\bar{f}(\hat{\pi})$ is a convex subset of S. The proof of upper hemicontinuity of f is similar to the proof in Theorem 1, so we leave it out. We now show that f is upper hemicontinuous implies $\bar{f}$ is upper hemicontinuous.

For all $\hat{\pi}^i \in S$, $i = 1,2,\ldots$ and $\hat{\pi}^i \to \hat{\pi}$, for all $(\pi^{*i}/e\pi^{*i}) \in \bar{f}(\hat{\pi}^i)$ and $(\pi^{*i}/e\pi^{*i}) \to \pi^*$.

Case 1. If there exists a subsequence $\pi^{*i}j \to k\pi^*$ where $k \in (0, \infty)$, from the upper hemicontinuity of f, we know $k\pi^* \in f(\hat{\pi})$, but $\pi^* = k\pi^*/(ke\pi^*) \in \bar{f}(\hat{\pi})$, therefore in this case $\bar{f}$ is upper hemicontinuous.

Case 2. If for all $k \in (0, +\infty)$, there does not exists $\pi^{*i}j \to k\pi^*$, then there exists a subsequence $\pi^{*i}j$, $j = 1, 2, \ldots,$ such that $e\pi^{*i}j \to +\infty$. Without lose of generality, assume $e\pi^{*i} \to +\infty$. Since $\pi^{*i}b - \sigma^{*i}K = -\theta^{*i}$ still holds, $\{\theta^{*i}, i = 1, 2, \ldots\}$ still bounded. Dividing the above equality by $e\pi^{*i}$, $(\pi^{*i}b/e\pi^{*i}) - (\sigma^{*i}K/e\pi^{*i}) = -(\theta^{*i}/e\pi^{*i})$. Letting $i \to \infty$, we have $\pi^*b - \lim_{i\to\infty}(\sigma^{*i}K/e\pi^{*i}) = 0$. Letting $\xi i = (\sigma^{*i}/e\pi^{*i})$, $\{\xi i\ i = 1, 2, \ldots\}$ is bounded. Without loss of generality, assume $\xi^i \to \xi$, then $\xi \geq 0$ and $\lim_{i\to\infty}(\sigma^{*i}K/e\pi^{*i}) = \xi \cdot K$. Dividing $\pi^{*i}A \leq \sigma^{*i}$ and $\pi^{*i}(-d^0 + M\hat{\pi}^i) \leq -1$ by $e\pi^{*i}$ and letting $i \to \infty$, we get $\pi^*A \leq \xi$, $\pi^*(-d^0 + M\hat{\pi}) \leq 0$. Because $\pi^* \neq 0$ and $(-d^0 + M\hat{\pi}) < 0$, there exists $k > 0$, such that $k\pi^*(-d^0 + M\hat{\pi}) < -1$, $k\pi^*A \leq k\xi$, $k\pi^* \geq 0$ $k\xi \geq 0$, $k\pi^*b - k\xi \cdot K = 0$. This means $(k\xi, k\pi^*)$ is

feasible for $(D\hat{\pi})$. On the other hand, $\{Y : AY \geq b, 0 \leq Y \leq K\} \neq \phi$, so there exists Y^*, such that $(Y^*, 0)$ is feasible for $(P\hat{\pi})$. Because $k\pi^* b - k\xi \cdot K = 0$, by weak duality theorem, $(k\xi, k\pi^*)$ is an optimal solution of $(D\hat{\pi})$ and $(Y^*, 0)$ is an optimal solution of $(P\hat{\pi})$, therefore $k\pi^* \in f(\hat{\pi})$, $\pi^* = [k\pi^*/e(k\pi^*)] \in \bar{f}(\hat{\pi})$, i.e., $\bar{f}$ is upper hemicontinuous. By Kakutani Fixed Point Theorem, $\bar{f}$ has a fixed point. Q.E.D.

Economic Interpretation:

Let A be the technology matrix of an economy and $Y \geq 0$ the level of production, $Y = (Y_1, Y_2, \ldots, Y_n)^T$. The net production available for consumption is AY. Let K be vector of capacities available so that $Y \leq K$. Consumption is a vector $b + \theta d$ where b is the fixed part, θ is a scalar, and d is the variable part that depends on relative prices. We assume, at fixed relative prices, the economy acts so as to maximize θ,

$$\begin{aligned} \max \quad & \theta \\ \text{s.t.} \quad & AY - (b + \theta d) \geq 0 \\ & 0 \leq Y \leq K \end{aligned}$$

We assume the variable demand d depends on relative prices π. Thus if $\pi = \hat{\pi}$, let us suppose $d = -(1/e\hat{\pi}) M\hat{\pi}$, where M is positive definite but not necessarily symmetric matrix.

We play a little game. We guess values for $\hat{\pi}$, compute d, solve the LP and determine optimal θ^*, Y^*, σ^*, π^*. Next we form (normalized π^*) $\pi^*/e\pi^*$ and compare it with (normalized $\hat{\pi}$) $\hat{\pi}/e\hat{\pi}$. If equal, the game is over, i.e., we have found an equilibrium price. If not, we guess

another $\hat{\pi}$ and try again. Question is: is there a choice of $\hat{\pi}$ such that $(\pi^*/e\pi^*) = (\hat{\pi}/e\hat{\pi})$? Under the assumption of Theorem 1 or Theorem 2, such equilibrium prices exist.

Finally we would like to point out that the theorems and proofs can be generalized to a n-period planning model.

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by Hui Hu

Consider parametric LP:

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The economic interpretation of the above model will be given at the end of the paper.

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